J.K.SHAH CLASSES

SYJC : 2016 – 17 **MATHEMATICS & STATISTICS** **DATE : 11/7/2016**

SOLUTION TO PAPER – 1 – SET 1

Q1.

01.

SOLUTION :

STEP 1

$\lim_{x \to 4} f(x)$

$= \lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

$= \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4}$ $x - 4 \neq 0$

$= \lim_{x \to 4} x + 4$

$= 4 + 4$

$= 8$

STEP 2 :

$f(4) = 8$ ……… given

STEP 3 :

$f(4) = \lim_{x \to 4} f(x)$

$\therefore f$ is continuous at $x = 4$

02.

SOLUTION :

STEP 1

$\lim_{x \to 1} f(x)$

$= \lim_{x \to 1} \frac{x^{12} - 1}{x - 1}$

$= \lim_{x \to 1} \frac{x^{12} - 1^{12}}{x - 1}$

$= 12(1)^{12 - 1}$

$= 12(1)^{11}$

$= 12$

STEP 2 :

$f(1) = k$ ……… given

STEP 3 :

Since f is continuous at $x = 1$

$f(1) = \lim_{x \to 1} f(x)$;

$k = 12$

03.

SOLUTION :

STEP 1

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \left(1 + \frac{5x}{2}\right)^{\frac{2}{x}}$

$= \lim_{x \to 0} \left(\left(1 + \frac{5x}{2}\right)^{\frac{2}{5x}}\right)^{5}$

$= e^{5}$

STEP 2 :

$f(0) = e^{5/2}$……… given

STEP 3 :

$f(0) \neq \lim_{x \to 0} f(x)$

$\therefore f$ is discontinuous at $x = 0$

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at $x = 0$ by redefining it as

$f(x) = \left(1 + \frac{5x}{2}\right)^{2/x}$; $x \neq 0$

$= e^{5}$; $x = 0$

04.

SOLUTION :

STEP 1

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x\to 0} \frac{\sin 9x}{2x}$$

$$= \lim_{x\to 0} \frac{9}{2} \frac{\sin 9x}{9x}$$

$$= \frac{9}{2}(1)$$

$$= \frac{9}{2}$$

STEP 2 :

$f(0) = 1/2$ ……… given

STEP 3 :

$$f(0) \neq \lim_{x\to 0} f(x)$$

$\therefore$ f is discontinuous at $x = 0$

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at $x = 0$ by redefining it as

$$f(x) = \frac{\sin 9x}{2x} \quad ; \quad x \neq 0$$

$$= 9/2 \quad ; \quad x = 0$$

05.

SOLUTION

$y = \tan^{-1}(\cot 2x)$

$y = \tan^{-1} \tan(\pi/2 - 2x)$

$y = \pi/2 - 2x$

Differentiate wrt x

$$\frac{dy}{dx} = 0 - 2$$

$$\frac{dy}{dx} = -2$$

06.

SOLUTION

Taking log on both sides

$\text{Log } y = \sin^{-1}x \,.\, \log x$

Differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = \sin^{-1}x \,.\, \frac{d}{dx}\log x + \log x \frac{d}{dx}\sin^{-1}x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin^{-1}x \,.\, \frac{1}{x} + \log x \,.\, \frac{1}{\sqrt{1-x^2}}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = y\left(\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)$$

07.

Solution

Put $x = \sin\theta$

$y = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$y = \sin^{-1} \sin 3\theta$

$y = 3\theta$

$y = 3\sin^{-1} x$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

08.

SOLUTION

$\mathbf{u = e^{4x + 5}}$

Diff wrt x

$$\frac{du}{dx} = e^{4x+5} \frac{d}{dx}(4x + 5)$$

$= e^{4x + 5} . 4$

$= 4.e^{4x + 5}$

$\mathbf{v = e^{3x}}$

Diff wrt x

$\frac{dv}{dx} = e^{3x} \frac{d}{dx}(3x)$

$= e^{3x} . 3$

$= 3.e^{3x}$

Now

$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$= \frac{4.e^{4x + 5}}{3.e^{3x}}$

$= \frac{4}{3}.e^{4x + 5 - 3x}$

$= \frac{4}{3}.e^{x + 5}$

Q2 . A

01.

SOLUTION :

STEP 1

$\lim_{x\to1-} f(x)$

$= \lim_{x\to1} 5x - 3$

$= 5(1) - 3 = 2$

STEP 2

$\lim_{x\to1+} f(x)$

$= \lim_{x\to1} x^2 + 1$

$= 1^2 + 1 = 2$

STEP 3

$f(1) = 1^2 + 1 = 2$

STEP 4

$\lim_{x\to1-} f(x) = \lim_{x\to1+} f(x) = f(1)$

f is continuous at $x = 1$

02.

SOLUTION :

STEP 1

$\lim_{x\to 0} f(x)$

$= \lim_{x\to0} \frac{\cos3x - \cos7x}{x^2}$

$= \lim_{x\to0} \frac{-2 \sin \frac{3x + 7x}{2} . \sin \frac{3x - 7x}{2}}{x^2}$

$= \lim_{x\to0} \frac{-2 \sin \frac{10x}{2} . \sin \frac{-4x}{2}}{x^2}$

$= \lim_{x\to0} \frac{-2 \sin 5x . \sin -2x}{x^2}$

$= \lim_{x\to0} \frac{2 \sin 5x . \sin 2x}{x^2}$

$= \lim_{x\to0} 2 \frac{\sin 5x}{x} . \frac{\sin 2x}{x}$

$= \lim_{x\to0} 2.5 \frac{\sin 5x}{5x} .2. \frac{\sin 2x}{2x}$

$= 2.5(1) . 2 (1)$

$= 20$

STEP 2 :

$f(0) = 10$……… given

STEP 3 :

$f(0) \neq \lim_{x\to0} f(x)$

$\therefore f$ is discontinuous at $x = 0$

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at x = 0 by redefining it as

$$f(x) = \frac{\cos 3x - \cos 7x}{x^2} \quad ; \quad x \neq 0$$
$$= 20 \quad ; \quad x = 0$$

03.

SOLUTION :

STEP 1

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{10^x - 5^x - 2^x + 1}{x^2}$$

$$= \lim_{x \to 0} \frac{(5.2)^x - 5^x - 2^x + 1}{x^2}$$

$$= \lim_{x \to 0} \frac{5^x.2^x - 5^x - 2^x + 1}{x^2}$$

$$= \lim_{x \to 0} \frac{5^x(2^x - 1) - 1(2^x - 1)}{x^2}$$

$$= \lim_{x \to 0} \frac{(5^x - 1).(2^x - 1)}{x^2}$$

$$= \lim_{x \to 0} \frac{5^x - 1}{x} . \frac{2^x - 1}{x}$$

$$= \log 5 . \log 2$$

STEP 2 :

$f(0) = \log 10$ ……… given

STEP 3 :

$$f(0) \neq \lim_{x \to 0} f(x)$$

$\therefore$ f is discontinuous at x = 0

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at x = 0 by redefining it as

$$f(x) = \frac{10^x - 5^x - 2^x + 1}{x^2} \quad ; \quad x \neq 0$$
$$= \log 5 . \log 2 \quad ; \quad x = 0$$

Q2 . B

01.

SOLUTION :

STEP 1

$$\lim_{x \to 0+} f(x)$$

$$= \lim_{x \to 0} x^2 + a$$

$$= 0^2 + a = a$$

STEP 2

$$\lim_{x \to 0-} f(x)$$

$$= \lim_{x \to 0} 2\sqrt{x^2 + 1} + b$$

$$= 2\sqrt{0^2 + 1} + b$$

$$= 2 + b$$

STEP 3

$$f(0) = 0^2 + a$$
$$= a$$

STEP 4

Since f is continuous at x = 0

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$$

$$2 + b = a = a$$

$$2 + b = a \quad \text{…………………………} \quad (1)$$

STEP 5

$f(1) = 2$

$1^2 + a = 2$ $\quad \therefore \quad a = 1$

Sub in (1)

$2 + b = 1$ $\quad \therefore \quad b = -1$

02.

SOLUTION :

STEP 1

$$\lim_{x \to 0+} f(x)$$

$$= \lim_{x \to 0} \frac{e^{2x} - 1}{ax}$$

$$= \lim_{x \to 0} \frac{1}{a} \frac{e^{2x} - 1}{x}$$

$$= \lim_{x \to 0} \frac{2}{a} \frac{e^{2x} - 1}{2x}$$

$$= \frac{2.\log e}{a}$$

$$= 2/a$$

STEP 2

$$\lim_{x \to 0-} f(x)$$

$$= \lim_{x \to 0} \frac{\log (1 + 7x)}{bx}$$

$$= \lim_{x \to 0} \frac{1}{bx} \log (1 + 7x)$$

$$= \lim_{x \to 0} \frac{1}{b} \log (1 + 7x)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \frac{1}{b} \log \left((1 + 7x)^{\frac{1}{7x}} \right)^7$$

$$= \frac{1}{b} \log e^7$$

$$= \frac{7}{b} \log e$$

$$= \frac{7}{b}$$

STEP 3

$f(0) = 1$ ……… given

STEP 4

Since f is continuous at $x = 0$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$$

$$\frac{2}{a} = \frac{7}{b} = 1$$

$\therefore$ $a = 2$ & $b = 7$

03.

SOLUTION :

STEP 1

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{x^2}{e^x + e^{-x} - 2}$$

$$= \lim_{x \to 0} \frac{x^2}{e^x + \frac{1}{e^x} - 2}$$

$$= \lim_{x \to 0} \frac{x^2}{\frac{(e^x)^2 + 1 - 2.e^x}{e^x}}$$

$$= \lim_{x \to 0} \frac{x^2}{\frac{(e^x - 1)^2}{e^x}}$$

$$= \lim_{x \to 0} \frac{e^x}{\frac{(e^x - 1)^2}{x^2}}$$

$$= \lim_{x \to 0} \frac{e^x}{\left(\frac{e^x - 1}{x}\right)^2}$$

$$= \frac{e^0}{(\log e)^2}$$

$$= 1$$

STEP 2

Since f is continuous at $x = 0$

$$f(0) = \lim_{x \to 0} f(x)$$

$$f(0) = 1$$

Q3. A

01.

SOLUTION

$$x^7y^9 = (x + y)^{16}$$

taking log on both sides

$$7 \log x + 9 \log y = 16 \log (x + y)$$

Differentiating wrt x

$$7 \frac{1}{x} + 9 \frac{1}{y} \frac{dy}{dx} = 16 \frac{1}{x + y} \frac{d}{dx}(x + y)$$

$$\frac{7}{x} + \frac{9}{y} \frac{dy}{dx} = \frac{16}{x + y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{7}{x} + \frac{9}{y} \frac{dy}{dx} = \frac{16}{x + y} + \frac{16}{x + y} \frac{dy}{dx}$$

$$\left(\frac{9}{y} - \frac{16}{x + y}\right) \frac{dy}{dx} = \frac{16}{x + y} - \frac{7}{x}$$

$$\frac{9x + 9y - 16y}{y.(x + y)} \frac{dy}{dx} = \frac{16x - 7x - 7y}{x(x + y)}$$

$$\frac{9x - 7y}{y.(x + y)} \frac{dy}{dx} = \frac{9x - 7y}{x(x + y)}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

02.

SOLUTION

$$y = \tan^{-1} \frac{x}{1 + 20x^2}$$

$$y = \tan^{-1} \left(\frac{5x - 4x}{1 + 5x.4x}\right)$$

$$y = \tan^{-1} 5x - \tan^{-1} 4x$$

$$\frac{dy}{dx} = \frac{1}{1 + 25x^2} . \frac{d}{dx}(5x) - \frac{1}{1 + 16x^2} . \frac{d}{dx}(4x)$$

$$\frac{dy}{dx} = \frac{5}{1 + 25x^2} - \frac{4}{1 + 16x^2}$$

03.

SOLUTION

$$y = \tan^{-1} \left(\frac{\frac{\sin 2x - \cos 2x}{\sin 2x}}{\frac{\sin 2x + \cos 2x}{\sin 2x}}\right)$$

$$y = \tan^{-1} \left(\frac{\frac{\sin 2x}{\sin 2x} - \frac{\cos 2x}{\sin 2x}}{\frac{\sin 2x}{\sin 2x} + \frac{\cos 2x}{\sin 2x}}\right)$$

$$y = \tan^{-1} \left(\frac{1 - \cot 2x}{1 + \cot 2x}\right)$$

$$y = \tan^{-1} 1 - \tan^{-1} \cot 2x$$

$$y = \tan^{-1} 1 - \tan^{-1} \tan \left(\frac{\pi}{2} - 2x\right)$$

$$y = \tan^{-1} 1 - \left(\frac{\pi}{2} - 2x\right)$$

$$y = \tan^{-1} 1 - \frac{\pi}{2} + 2x$$

$$\frac{dy}{dx} = 0 + 0 + 2$$

$$\frac{dy}{dx} = 2$$

Q3 B

01.

SOLUTION

$\sin y = x.\sin(5 + y)$

$x = \dfrac{\sin y}{\sin(5 + y)}$

Differentiating wrt y

$$\frac{dx}{dy} = \frac{\sin(5 + y)\,\frac{d}{dy}\sin y - \sin y\,\frac{d}{dy}\sin(5 + y)}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin(5 + y)\,.\,\cos y - \sin y\,.\,\cos(5 + y)\,\frac{d}{dy}(5 + y)}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin(5 + y).\cos y - \cos(5 + y)\,.\,\sin y}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin(5 + y - y)}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin 5}{\sin^2(5 + y)}$$

cont.

Now $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\sin^2(5+ y)}{\sin 5}$

02.

Solution

$\mathbf{x = \dfrac{2t}{1 + t^2}}$

Put $t = \tan\theta$

$x = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

$x = \sin 2\theta$ ………. (1)

diff wrt 'θ'

$\dfrac{dx}{d\theta} = \cos 2\theta\ .\ \dfrac{d}{d\theta}2\theta$

$= \cos 2\theta\ .\ 2$

$= 2\cos 2\theta$

$\mathbf{y = \dfrac{1 - t^2}{1 + t^2}}$

Put $t = \tan\theta$

$y = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$y = \cos 2\theta$ ………. (2)

diff wrt 'θ'

$\dfrac{dy}{d\theta} = -\sin 2\theta\ .\ \dfrac{d}{d\theta}2\theta$

$= -\sin 2\theta\ .\ 2$

$= -2\sin 2\theta$

Now

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$\dfrac{dy}{dx} = \dfrac{-2\sin 2\theta}{2\cos 2\theta}$

$\dfrac{dy}{dx} = -\dfrac{\sin 2\theta}{\cos 2\theta}$

$\dfrac{dy}{dx} = -\dfrac{x}{y}$

03.

$y = u + v$; where

$u = (\sin x)^x$

Taking log on both sides

$\text{Log } y = x \cdot \log(\sin x)$

Differentiating wrt x

$$\frac{1}{u}\frac{du}{dx} = x \frac{d}{dx}\log(\sin x) + \log(\sin x)\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx} = x \frac{1}{\sin x}\frac{d}{dx}\sin x + \log(\sin x) \cdot 1$$

$$\frac{1}{u}\frac{du}{dx} = x \frac{1}{\sin x} \cdot \cos x + \log(\sin x)$$

$$\frac{1}{u}\frac{du}{dx} = x.\cot x + \log(\sin x)$$

$$\frac{du}{dx} = u\left[x.\cot x + \log(\sin x)\right]$$

cont.

$$\frac{du}{dx} = (\sin x)^x\left[x.\cot x + \log(\sin x)\right] \quad \ldots\ldots \quad (2)$$

$v = x \cdot \sin x$

Differentiating wrt x

$$\frac{dv}{dx} = x \cdot \frac{d}{dx}\sin x + \sin x \frac{d}{dx} x$$

$$\frac{dv}{dx} = x.\cos x + \sin x \quad \ldots\ldots. \quad (3)$$

Hence

$$\frac{dy}{dx} = (\sin x)^x\left[x.\cot x + \log(\sin x)\right] + x.\cos x + \sin x$$